

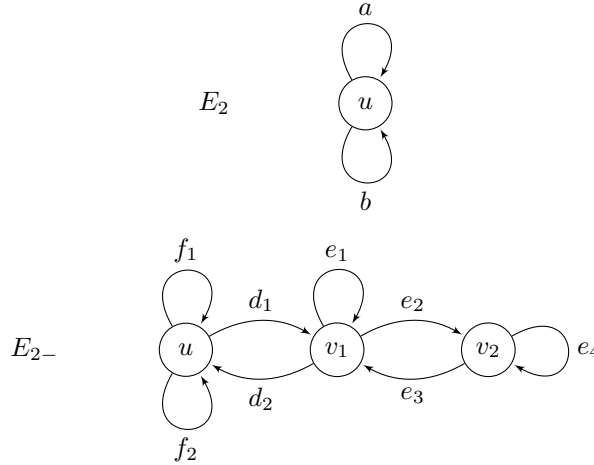
THE CUNTZ SPLICE DOES NOT PRESERVE *-ISOMORPHISM OF LEAVITT PATH ALGEBRAS OVER \mathbb{Z}

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ABSTRACT. We show that the Leavitt path algebras $L_{2,\mathbb{Z}}$ and $L_{2-,\mathbb{Z}}$ are not isomorphic as $*$ -algebras. There are two key ingredients in the proof. One is a partial algebraic translation of Matsumoto and Matui's result on diagonal preserving isomorphisms of Cuntz–Krieger algebras. The other is a complete description of the projections in $L_{\mathbb{Z}}(E)$ for E a finite graph. This description is based on a generalization, due to Chris Smith, of the description of the unitaries in $L_{2,\mathbb{Z}}$ given by Brownlowe and the second named author. The techniques generalize to a slightly larger class of rings than just \mathbb{Z} .

1. INTRODUCTION

How are two Leavitt path algebras related if the underlying graphs are related by the so called Cuntz splice? This is arguably one of the most important open problems in the theory of graph algebras. More concretely, the question is whether the Cuntz splice preserves either the Morita equivalence class or the isomorphism class of the Leavitt path algebras considered as either rings, algebras, or $*$ -algebras. In this paper, we provide the first answer to any of these questions: We show that there cannot exist a $*$ -isomorphism between the Leavitt path algebras over the ring of integers induced by the graphs E_2 and E_{2-} shown below.



Some authors prefer the name R_2 (“rose with two petals”) for the graph we call E_2 .

The interest in isomorphism and Morita equivalence of Leavitt path algebras of graphs related by the Cuntz splice stems from a desire to understand whether Leavitt path algebras can be classified through algebraic analogues of the celebrated results of Rørdam ([36]) and Kirchberg–Phillips ([24, 25, 33]). We now give a short overview of some of the historic developments in the field.

To each irreducible non-permutation matrix $A \in M_n(\mathbb{N})$, one can associate a shift of finite type, \mathcal{X}_A . For such shift spaces, the Bowen–Franks invariant consisting of the Bowen–Franks group $BF(A) = \mathbb{Z}^n / (I - A)\mathbb{Z}^n$ and the sign of the determinant of $I - A$ is a complete invariant of flow equivalence [13, 22, 32]. To each non-permutation matrix $A \in M_n(\mathbb{N})$, one can also associate a C^* -algebra \mathcal{O}_A , called the Cuntz–Krieger algebra, and the stable isomorphism class of \mathcal{O}_A is a flow invariant for \mathcal{X}_A . In fact, $K_0(\mathcal{O}_A) \cong BF(A)$. Considering the flow classification of \mathcal{X}_A , one might wonder if \mathcal{O}_A also contains enough information to determine the other half of the Bowen–Franks invariant, $\text{sgn}(\det(I - A))$. On its own, \mathcal{O}_A does not: Rørdam proved that simple Cuntz–Krieger algebras are classified up to stable isomorphism by their K_0 -groups [36].

Leavitt path algebras were introduced independently in [3] and [10] as universal algebras over some field K with relations given by the underlying graph. They are purely algebraic analogues of graph C^* -algebras, which, in turn, are generalizations of Cuntz–Krieger algebras, see for instance [12, 20, 21, 26, 27, 34]. More generally, work of Tomforde shows how to define a Leavitt path algebra over any unital, commutative ring [40]. A good overview of the theory of Leavitt path algebras can be found in the survey paper [1].

In [5], it is shown that if \mathcal{X}_A is flow equivalent to \mathcal{X}_B for finite irreducible non-permutation matrices A and B , then $L_K(E_A)$ is Morita equivalent to $L_K(E_B)$ for any field K . It is also shown that the K_0 -group of $L_K(A)$ coincides with the Bowen–Franks group of A , so that a Leavitt path algebra is classified by its K_0 -group and the sign of the determinant. However, it is an open problem whether the sign of the determinant is an isomorphism invariant, and this leaves the theory of Leavitt path algebras in a situation very similar to the situation in the theory of Cuntz–Krieger algebras prior to Rørdam’s classification result.

For the graphs E_2 and $E_{2,-}$ defined above, it is straightforward to check that the associated Bowen–Franks groups are trivial. However, the signs of the determinants are different, and hence, the associated shifts of finite type are not flow equivalent. The graph $E_{2,-}$ is constructed from E_2 by attaching two extra vertices and the associated edges. This gluing operation is called “performing a Cuntz splice”, and it has played an important role in the study Cuntz–Krieger algebras, graph C^* -algebras, and Leavitt path algebras [5, 19, 35, 36, 37, 38]. It serves to flip the sign of the determinant while keeping the Bowen–Franks group fixed. The graph E_2 is the simplest possible example of a graph where this operation can be performed without obviously changing Morita equivalence class.

In Rørdam’s work, it was shown that $C^*(E_2) \cong C^*(E_{2,-})$, and this was then used to derive the general non-reliance on the sign of the determinant [36]. If we wish to pursue a similar approach to the classification of Leavitt path algebras and understand whether there can exist an algebraic Kirchberg–Phillips theorem in this setting, then we must know whether the Leavitt path algebras associated to E_2 and $E_{2,-}$ are isomorphic.

Our main results are Theorem 3.6, Theorem 6.3, and Corollary 5.7. All these results concern a subalgebra of $L_R(E)$ called the diagonal, see Definition 2.7. Specifically, Theorem 3.6 states that for a subring R of \mathbb{C} that is closed under complex conjugation and graphs E and F satisfying standard assumptions, there can only exist a diagonal preserving $*$ -isomorphism between $L_R(E)$ and $L_R(F)$ if the Bowen–Franks determinants are equal. This is a partial algebraic analogue of a result of Matsumoto and Matui ([30]) and it is derived directly from their result. Theorem 6.3 shows that if all projections in $L_R(F)$ are elements of the diagonal, then $L_R(E)$

and $L_R(F)$ can only be isomorphic as $*$ -algebras if their Bowen–Franks determinants are equal. Corollary 5.7 shows that all projections of $L_{\mathbb{Z}}(E)$ are elements of the diagonal if E is a finite graph.

In the initial version of this paper, we only proved that all projections in $L_{\mathbb{Z}}(E_2)$ are diagonal, which was sufficient to let us conclude that $L_{\mathbb{Z}}(E_2)$ and $L_{\mathbb{Z}}(E_2-)$ are not $*$ -isomorphic. Our arguments were based on a complete description of the unitaries in $L_{\mathbb{Z}}(E_2)$ given in [15]. Since then, Chris Smith has shown us how to extend the description of the unitaries in $L_{\mathbb{Z}}(E_2)$ from [15] to cover all finite graphs, see Proposition 4.4, and we have realized how to extend this from \mathbb{Z} to slightly more general rings. Therefore, we can show that all projections in $L_R(E)$ are diagonal whenever E is a finite graph and R is a unital, commutative subring of \mathbb{C} closed under complex conjugation that has an essentially unique partition of the unit (see Definition 2.8). In particular, this holds if $R = \mathbb{Z}$.

The three main results combine to show that if E, F are finite, strongly connected, essential and non-trivial graphs (see Definition 2.3), if R is a unital, commutative subring of \mathbb{C} closed under complex conjugation that has an essentially unique partition of the unit and if $L_R(E)$ and $L_R(F)$ are $*$ -isomorphic, then $\text{sgn}(I - A_E) = \text{sgn}(I - A_F)$ (Corollary 6.4). Possible applications of the results to questions concerning Morita equivalence and Leavitt path algebras over fields are discussed in Remarks 6.6 and 6.7.

Since the initial version of this paper appeared on the arXiv, there have been some interesting developments related to the problem discussed here. In [7, Corollary 4.8] the authors use groupoid techniques to show that when R is an integral domain, there can be no diagonal preserving *ring* isomorphism from $L_{2,R}$ to $L_{2-,R}$. In [16, Corollary 4.7], it is shown that $L_{2,\mathbb{Z}}$ to $L_{2-,\mathbb{Z}}$ are not stably isomorphic as $*$ -algebras, in a suitable sense.

2. PRELIMINARIES

Definition 2.1. A graph $E = (E^0, E^1, r, s)$ is a four tuple where E^0 is the vertex set, E^1 the edge set, and $r, s: E^1 \rightarrow E^0$ are the range and source maps.

Given a graph E , we denote by A_E its adjacency matrix, i.e. the $E^0 \times E^0$ matrix with $A(v, w) = |s^{-1}(v) \cap r^{-1}(w)|$. The following graph concepts will be needed below.

Definition 2.2 ([26]). A graph E satisfies Condition (L) if every cycle in E has an exit.

Definition 2.3. A graph E is said to be:

- *strongly connected* if there is a path between any two vertices,
- *essential* if it has no sinks or sources, and
- *trivial* if it is a single cycle with no other edges or vertices.

Note that if E is an essential and strongly connected graph then A_E will be an irreducible matrix with no zero rows or columns. Furthermore, if E is non-trivial then A_E will not be a permutation matrix.

For a graph E , E^n will denote the set of *paths* of length n , and $E^* = \cup_{n \in \mathbb{N}} E^n$ will denote the collection of all finite paths. The range and source maps are extended to these sets in the natural way. Similarly, $E^\infty = \{e_1 e_2 \cdots \mid e_i \in E^1 \text{ and } r(e_i) = s(e_{i+1})\}$ will denote the set of infinite paths. In the case of E_2 , $E_2^* = \{a, b\}^*$ and $E_2^\infty = \{a, b\}^\mathbb{N}$. The *cylinder set* of $\alpha \in E^*$ is defined to be

$$\mathcal{Z}(\alpha) = \{\alpha\xi \mid \xi \in E^\infty \text{ and } \alpha\xi \in E^\infty\}.$$

For $\alpha, \beta \in E^*$, $\mathcal{Z}(\alpha)$ and $\mathcal{Z}(\beta)$ are disjoint if and only if neither α nor β extends the other.

To each graph E , we associate a graph C^* -algebra and a collection of Leavitt path algebras as defined below.

Definition 2.4 ([21, Definition 1]). Let E be a graph. The graph C^* -algebra of E , $C^*(E)$, is the universal C^* -algebra generated by mutually orthogonal projections $\{p_v \mid v \in E^0\}$ and partial isometries $\{s_e \mid e \in E^1\}$ subject to the relations

- (i) $s_e^* s_f = 0$, if $e \neq f$,
- (ii) $s_e^* s_e = p_{r(e)}$,
- (iii) $s_e s_e^* \leq p_{s(e)}$, and,
- (iv) $p_v = \sum_{e \in s^{-1}(v)} s_e s_e^*$, if $s^{-1}(v)$ is finite and nonempty.

Readers unfamiliar with the subject should be warned that there are two competing conventions for the definition of $C^*(E)$. This dichotomy stems from a necessary asymmetry between sinks and sources in the defining relations. Raeburn's monograph [34] uses the other possible convention.

Definition 2.5 ([40, Definitions 2.4 and 3.4]). Let R be a commutative ring with unit and let E be a graph. The Leavitt path algebra of E over R is the universal R -algebra generated by pairwise orthogonal idempotents $\{v \mid v \in E^0\}$ and elements $\{e, e^* \mid e \in E^1\}$ satisfying

- (i) $e^* f = 0$, if $e \neq f$,
- (ii) $e^* e = r(e)$,
- (iii) $s(e)e = e = er(e)$,
- (iv) $e^* s(e) = e^* = r(e)e^*$, and,
- (v) $v = \sum_{e \in s^{-1}(v)} ee^*$, if $s^{-1}(v)$ is finite and nonempty.

By [40, Proposition 3.4], $L_R(E) = \text{span}_R\{\alpha\beta^* \mid \alpha, \beta \in E^*, r(\alpha) = r(\beta)\}$. For any unital, commutative ring R , we can extend the map $\alpha\beta^* \mapsto \beta\alpha^*$ to a *linear* involution on $L_R(E)$. If R is a subring of \mathbb{C} that is closed under complex conjugation, we will extend $\alpha\beta^* \mapsto \beta\alpha^*$ to a *conjugate linear* involution on $L_R(E)$ instead. Throughout this paper, we are mainly interested in the latter case.

Since we consider $L_R(E)$ as a $*$ -algebra, we prefer to work with self-adjoint idempotents rather than just idempotents. Following the name conventions of C^* -algebras, we call these elements projections.

Definition 2.6. An element $p \in L_R(E)$ is called a *projection* if $p = p^2 = p^*$.

Each graph algebra contains a distinguished subalgebra called the diagonal:

Definition 2.7. Let E be a graph and R a commutative ring with unit. We define

$$\mathcal{D}(C^*(E)) = \overline{\text{span}}_{\mathbb{C}}\{s_\alpha s_\alpha^* \mid \alpha \in E^*\},$$

and

$$\mathcal{D}(L_R(E)) = \text{span}_R\{\alpha\alpha^* \mid \alpha \in E^*\}.$$

We refer to these sets as the diagonal of $C^*(E)$ and $L_R(E)$, respectively.

It is shown in [23, Theorem 5.2] and [31, Theorem 3.7] that $\mathcal{D}(C^*(E))$ is a MASA (maximal abelian sub-algebra) in $C^*(E)$ if E satisfies Condition (L). We note that $\mathcal{D}(L_R(E))$ is generated by projections. It has recently been shown that if E satisfies condition (L) then $\mathcal{D}(L_R(E))$ is maximal abelian in $L_R(E)$ (see [17, Proposition 3.12 and Theorem 3.9] and [7, Lemma 3.13]).

We are interested in subrings of \mathbb{C} that behave like \mathbb{Z} in the following very specific way.

Definition 2.8 (Essentially unique partition of the unit). A unital (same unit) subring $R \subset \mathbb{C}$ closed under complex conjugation has an *essentially unique partition of the unit* if whenever $\lambda_1, \lambda_2, \dots, \lambda_n \in R$ satisfy

$$\sum_{i=1}^n \lambda_i \overline{\lambda_i} = \sum_{i=1}^n |\lambda_i|^2 = 1,$$

then all but one of the λ_i is zero.

Example 2.9. *The following is an incomplete list of subrings of \mathbb{C} with an essentially unique partition of the unit.*

- \mathbb{Z} .
- $\mathbb{Z} + \sqrt{p}\mathbb{Z}$, for p a prime.
- The Gaussian integers, $\mathbb{Z} + i\mathbb{Z}$.
- The ring $\mathbb{Z}[\pi]$, $\mathbb{Z} + \pi\mathbb{Z} + (\pi^2)\mathbb{Z} + \dots$.

Remark 2.10. Suppose $R \subset \mathbb{C}$ has an essentially unique partition of the unit. For each $n \neq 1$, $\frac{1}{n} \notin R$ since

$$\sum_{k=1}^{n^2} \left(\frac{1}{n}\right)^2 = 1.$$

Note that this means that no subfield of \mathbb{C} has an essentially unique partition of the unit.

3. MATSUMOTO AND MATUI'S RESULT FOR GRAPH ALGEBRAS

In [30], Matsumoto and Matui show that under standard assumptions on matrices A and B , there can only exist a diagonal preserving isomorphism between the Cuntz–Krieger algebras (see Definition 3.1) \mathcal{O}_A and \mathcal{O}_B if $\text{sgn}(\det(I - A)) = \text{sgn}(\det(I - B))$. In this section, we will translate this part of Matsumoto and Matui's spectacular result into the world of graph algebras. In Section 3.1, the result is translated into a statement about graph C^* -algebras and in Section 3.2, this is used to prove an algebraic analogue for Leavitt path algebras over subrings of \mathbb{C} . Both results are straightforward adaptations of those presented in [30].

3.1. Graph C^* -algebras. First, recall the definition of a Cuntz–Krieger algebra and its diagonal (originally from [18]).

Definition 3.1. Let A be an $N \times N$ matrix with entries in $\{0, 1\}$. The Cuntz–Krieger algebra \mathcal{O}_A is the universal C^* -algebra generated by partial isometries S_1, S_2, \dots, S_N satisfying the relations

- $\sum_{i=1}^N S_i S_i^* = 1$, and,
- $S_i^* S_i = \sum_{j=1}^N A(i, j) S_j S_j^*$,

where $A(i, j)$ denotes the ij 'th entry of A . The diagonal of \mathcal{O}_A is

$$\mathcal{D}(\mathcal{O}_A) = \overline{\text{span}}_{\mathbb{C}} \{S_{i_1} S_{i_2} \dots S_{i_k} S_{i_k}^* \dots S_{i_1}^* \mid k \in \mathbb{N}, i_1, i_2, \dots, i_k \in \{1, 2, \dots, N\}\}.$$

We observe that for two indices i, j we have

$$S_i S_j = S_i S_i^* S_i S_j = S_i \left(\sum_{k=1}^N A(i, k) S_k S_k^* \right) S_j = A(i, j) S_i S_j.$$

Hence, $S_i S_j = 0$ if $A(i, j) = 0$.

Condition (L) for graphs was designed to be an equivalent of Cuntz and Krieger's Condition (I) ([18]). We recall that a $\{0, 1\}$ matrix will satisfy Condition (I) if it is not a permutation matrix.

Graph C^* -algebras were originally defined to be generalizations of Cuntz–Krieger algebras, and there are strong connections between the two concepts. We have been unable to find a reference for the following lemma, but it is certainly well-known to researchers in the field.

Lemma 3.2. *Let A be an irreducible $N \times N$ matrix with entries in $\{0, 1\}$ that satisfies Condition (I). Let E_A be the graph with $E_A^0 = \{1, 2, \dots, N\}$, $E_A^1 = \{[ij] \mid A(i, j) = 1\}$ and range and source maps given by*

$$r([ij]) = j \quad \text{and} \quad s([ij]) = i.$$

There exists an isomorphism $\phi: C^(E_A) \rightarrow \mathcal{O}_A$ such that $\phi(\mathcal{D}(C^*(E_A))) = \mathcal{D}(\mathcal{O}_A)$.*

Proof. Define ϕ on generators by

$$\begin{aligned} \phi(p_i) &= S_i S_i^*, \\ \phi(s_{[ij]}) &= S_i S_j S_j^*. \end{aligned}$$

By (the proof of) [29, Proposition 4.1], ϕ is an isomorphism.

Given a path $\alpha = [i_1 i_2][i_2 i_3] \dots [i_{k-1} i_k]$ in E_A we see that

$$\begin{aligned} \phi(s_\alpha) &= (S_{i_1} S_{i_2} S_{i_2}^*)(S_{i_2} S_{i_3} S_{i_3}^*) \dots (S_{i_{k-1}} S_{i_k} S_{i_k}^*) \\ &= S_{i_1} (S_{i_2} S_{i_2}^* S_{i_2})(S_{i_3} S_{i_3}^* S_{i_3}) \dots (S_{i_{k-1}} S_{i_{k-1}}^* S_{i_{k-1}}) S_{i_k} S_{i_k}^* \\ &= S_{i_1} S_{i_2} \dots S_{i_k} S_{i_k}^*. \end{aligned}$$

Hence

$$\phi(s_\alpha s_\alpha^*) = S_{i_1} S_{i_2} \dots S_{i_k} S_{i_k}^* S_{i_k} S_{i_k}^* S_{i_{k-1}}^* \dots S_{i_1}^* = S_{i_1} S_{i_2} \dots S_{i_k} S_{i_k}^* S_{i_{k-1}}^* \dots S_{i_1}^*.$$

From this, it follows that $\phi(\mathcal{D}(C^*(E_A))) \subseteq \mathcal{D}(\mathcal{O}_A)$. Since $S_i S_j \neq 0$ if and only if $[ij]$ is an edge in E_A , the computation above also shows that ϕ maps $\mathcal{D}(C^*(E_A))$ onto $\mathcal{D}(\mathcal{O}_A)$. \square

Theorem 3.3 (cf. [30, Theorem 3.6]). *Let E, F be finite, essential, strongly connected graphs that satisfy Condition (L). If there exists an isomorphism $\phi: C^*(E) \rightarrow C^*(F)$ such that $\phi(\mathcal{D}(C^*(E))) = \mathcal{D}(C^*(F))$, then*

$$\text{sgn}(\det(I - A_E)) = \text{sgn}(\det(I - A_F)).$$

Proof. In order to apply the result from [30], out-splitting will be used to replace E and F by graphs without multiple edges (see [11, Section 3]). For each $v \in E^0$, partition $s^{-1}(v)$ into singleton sets. Note that these partitions are proper, and let E_1 be the graph obtained from the corresponding out-splitting. Note that since E is essential, strongly connected and satisfies Condition (L), E_1 will be essential, strongly connected and will satisfy Condition (L). Analogously, let F_1 be the graph obtained from F by such a complete out-splitting. By [14, Corollary 6.2], there exist diagonal preserving isomorphisms $C^*(E) \rightarrow C^*(E_1)$ and $C^*(F) \rightarrow C^*(F_1)$. Hence, there also exists a diagonal preserving isomorphism from $C^*(E_1)$ to $C^*(F_1)$.

Since E_1 and F_1 have no multiple edges, satisfy Condition (L), are essential, and strongly connected, A_{E_1} and A_{F_1} are irreducible $\{0, 1\}$ matrices that satisfy Condition (I). By Lemma 3.2, $C^*(E_1) \cong \mathcal{O}_{A_{E_1}}$ and $C^*(F_1) \cong \mathcal{O}_{A_{F_1}}$ in a diagonal preserving way, so the argument above guarantees the existence of a diagonal preserving isomorphism from $C^*(\mathcal{O}_{A_{E_1}})$ to $C^*(\mathcal{O}_{A_{F_1}})$, and hence, it follows from [30, Theorem 3.6] that $\text{sgn}(\det(I - A_{E_1})) = \text{sgn}(\det(I - A_{F_1}))$. For edge shifts, out-splitting produces a shift space conjugate to the original (see e.g. [28, Theorem 2.4.10]). Since flow equivalence is a coarser equivalence relation than conjugacy, this implies that the edge shift of E is flow equivalent to the edge shift of E_1 . Clearly,

the same relation exists between F and F_1 . By [32], the sign of the determinant is an invariant of flow equivalence, so

$$\operatorname{sgn}(\det(I - A_E)) = \operatorname{sgn}(\det(I - A_{E_1})) = \operatorname{sgn}(\det(I - A_{F_1})) = \operatorname{sgn}(\det(I - A_F)). \quad \square$$

3.2. Leavitt path algebras over certain subrings of \mathbb{C} . In this section, we will prove an algebraic analogue of Theorem 3.3, i.e. a weak algebraic version of Matsumoto and Matui's result. We will look only at rings R that are subrings of \mathbb{C} closed under complex conjugation, and we will equip $L_R(E)$ with a *conjugate* linear involution. In the following, \mathbb{Z} will be the main example of such a ring. First, we formalize the connection between $L_R(E)$ and $C^*(E)$.

Lemma 3.4. *Let E be a graph and let R be a subring of \mathbb{C} closed under complex conjugation. The map $\iota_E: L_R(E) \rightarrow C^*(E)$ given on generators by*

$$\begin{aligned} \iota_E(v) &= p_v, & \text{for } v \in E^0, \\ \iota_E(e) &= s_e, & \text{for } e \in E^1. \end{aligned}$$

*extends to a *-algebra embedding.*

Proof. By the universal property of $L_R(E)$, ι_E extends to a *-homomorphism. It is injective by the Graded Uniqueness Theorem ([40, Theorem 5.3]), see [39, Theorem 7.3] for details on how to apply the Graded Uniqueness Theorem. \square

We now provide a version of [6, Theorem 4.4] for Leavitt path algebras over subrings of \mathbb{C} (the result in [6] is for Leavitt path algebras over \mathbb{C}). This version of the result additionally tracks the image of the diagonal.

Lemma 3.5. *Let R be a subring of \mathbb{C} closed under complex conjugation and let E, F be finite graphs that satisfy Condition (L). If $\phi: L_R(E) \rightarrow L_R(F)$ is a *-isomorphism such that $\phi(\mathcal{D}(L_R(E))) \subseteq \mathcal{D}(L_R(F))$, then ϕ extends to an isomorphism $\bar{\phi}: C^*(E) \rightarrow C^*(F)$ such that $\bar{\phi}(\mathcal{D}(C^*(E))) = \mathcal{D}(C^*(F))$.*

Proof. As in the proof of [6, Theorem 4.4], ϕ extends to an isomorphism $\bar{\phi}: C^*(E) \rightarrow C^*(F)$ satisfying $\bar{\phi} \circ \iota_E = \iota_F \circ \phi$. Hence,

$$\begin{aligned} \bar{\phi}(\mathcal{D}(C^*(E))) &= \overline{\operatorname{span}}_{\mathbb{C}} \{ \bar{\phi}(s_{\alpha} s_{\alpha}^*) \mid \alpha \in E^* \} \\ &= \overline{\operatorname{span}}_{\mathbb{C}} \{ \bar{\phi}(\iota_E(\alpha \alpha^*)) \mid \alpha \in E^* \} \\ &= \overline{\operatorname{span}}_{\mathbb{C}} \{ \iota_F(\phi(\alpha \alpha^*)) \mid \alpha \in E^* \} \\ &= \overline{\operatorname{span}}_{\mathbb{C}} \{ \iota_F(\phi(\operatorname{span}_R \{ \alpha \alpha^* \mid \alpha \in E^* \})) \} \\ &= \overline{\operatorname{span}}_{\mathbb{C}} \{ \iota_F(\phi(\mathcal{D}(L_R(E)))) \} \\ &\subseteq \overline{\operatorname{span}}_{\mathbb{C}} \{ \iota_F(\mathcal{D}(L_R(F))) \} \\ &= \overline{\operatorname{span}}_{\mathbb{C}} \{ \operatorname{span}_R \{ s_{\alpha} s_{\alpha}^* \mid \alpha \in F^* \} \} \\ &= \overline{\operatorname{span}}_{\mathbb{C}} \{ s_{\alpha} s_{\alpha}^* \mid \alpha \in F^* \} \\ &= \mathcal{D}(C^*(F)). \end{aligned}$$

Since $\bar{\phi}$ is an isomorphism and $\mathcal{D}(C^*(E))$ is a MASA in $C^*(E)$, $\bar{\phi}(\mathcal{D}(C^*(E)))$ is a MASA in $C^*(F)$. As $\mathcal{D}(C^*(F))$ is also a MASA in $C^*(F)$, it follows that $\bar{\phi}(\mathcal{D}(C^*(E))) = \mathcal{D}(C^*(F))$. \square

Theorem 3.6. *Let E, F be finite, essential, strongly connected graphs that satisfy Condition (L), and let R be a subring of \mathbb{C} closed under complex conjugation. If $\phi: L_R(E) \rightarrow L_R(F)$ is a *-isomorphism such that $\phi(\mathcal{D}(L_R(E))) \subseteq \mathcal{D}(L_R(F))$, then*

$$\operatorname{sgn}(\det(I - A_E)) = \operatorname{sgn}(\det(I - A_F)).$$

Proof. By Lemma 3.5, ϕ extends to a diagonal preserving isomorphism $\bar{\phi}: C^*(E) \rightarrow C^*(F)$, so the result follows from Theorem 3.3. \square

4. UNITARIES IN $L_R(E)$

In this section, we show that all unitaries in $L_R(E)$ can be written in a certain standard form when E is a finite graph and R is a unital, commutative subring of \mathbb{C} closed under complex conjugation that has an essentially unique partition of the unit (see Proposition 4.4). This is similar to what is done in [15] for unitaries in $L_{\mathbb{Z}}(E_2)$, and the authors are grateful to Chris Smith for showing them how to generalize the arguments given there. A particularly interesting feature of Smith's argument is that it eschews the presentation of $L_{\mathbb{Z}}(E)$ as endomorphisms on an uncountably generated free module used in [15] in favor of techniques based on the grading of Leavitt path algebras. We generalize Smith's argument to cover all rings that have an essentially unique partition of the unit.

We first note a simple way to find unitaries in $L_R(E)$.

Lemma 4.1. *Let E be a finite graph and let R be a unital commutative ring with characteristic 0. If $\alpha_1, \alpha_2, \dots, \alpha_n \in E^*$ satisfy*

$$\sum_{i=1}^n \alpha_i \alpha_i^* = 1,$$

then $\alpha_i^ \alpha_j = 0$ for $i \neq j$.*

Proof. By relabeling, we can assume that $|\alpha_1| \geq |\alpha_2| \geq \dots \geq |\alpha_n|$. We see that

$$\alpha_1 = 1\alpha_1 = \left(\sum_{i=1}^n \alpha_i \alpha_i^* \right) \alpha_1 = \sum_{i=1}^n \alpha_i \alpha_i^* \alpha_1.$$

The first term in the sum on the right hand side is α_1 , and since α_1 has maximal length among the α_i , each subsequent term is either a real path or equals 0. Since the real paths are linearly independent ([40, Proposition 4.9]) and R has characteristic 0, it follows that $\alpha_i \alpha_i^* \alpha_1 = 0$, for each $i \neq 1$. Hence, $\alpha_i^* \alpha_1 = 0$. By taking adjoints, $\alpha_1^* \alpha_i = 0$ for $i = 2, 3, \dots, n$.

Let $k = 2, 3, \dots, n$ be given, and assume that $\alpha_i^* \alpha_k = 0$ for all $i < k$. Then

$$\alpha_k = 1\alpha_k = \sum_{i=1}^n \alpha_i \alpha_i^* \alpha_k = \sum_{i=k}^n \alpha_i \alpha_i^* \alpha_k.$$

Since α_k has maximal length among the α_i appearing in the sum on the right hand side, we see as above that $\alpha_i^* \alpha_k = 0 = \alpha_k^* \alpha_i$ for $i = k+1, k+2, \dots, n$. By induction, $\alpha_i^* \alpha_j = 0$ all $i \neq j$. \square

This gives us a simple way to find unitaries in $L_R(E)$:

Lemma 4.2. *Let E be a finite graph and let R be a unital commutative ring with characteristic 0. Suppose that $\alpha_i, \beta_i \in E^*$ and $\lambda_i \in R$, $i = 1, 2, \dots, n$, are such that*

- (1) $\sum_{i=1}^n \alpha_i \alpha_i^* = 1$,
- (2) $\sum_{i=1}^n \beta_i \beta_i^* = 1$,

and (if $L_R(E)$ is equipped with a linear involution)

- (3) $\lambda_i^2 = 1$ for all i

or (if $L_R(E)$ is equipped with a conjugate linear involution)

- (3) $\lambda_i \overline{\lambda_i} = 1$ for all i .

Then

$$u = \sum_{i=1}^n \lambda_i \alpha_i \beta_i^*,$$

is a unitary.

Proof. This follows from a simple computation of uu^* and u^*u using Lemma 4.1. \square

In general, not all unitaries in $L_R(E)$ will have the form described in Lemma 4.2. However, in the special case of $R = \mathbb{Z}$ all unitaries do in fact have this form. Before proving this result, we note the following simple consequence of the defining relations of a Leavitt path algebra.

Remark 4.3. Let E be a finite graph. Fix a vertex $v \in E^0$, some natural number m , and define

$$X_{v,m} = \{\gamma \in E^* \mid s(\gamma) = v \text{ and either } |\gamma| = m \text{ or } |\gamma| < m \text{ and } r(\gamma) \text{ is a sink}\}.$$

A straightforward induction argument using (v) from Definition 2.5 shows that

$$\sum_{\gamma \in X_{v,m}} \gamma \gamma^* = v.$$

Consider

$$x = \sum_{i=1}^n \lambda_i \alpha_i \beta_i^* \in L_R(E),$$

with $\lambda_i \in R$ and $r(\alpha_i) = r(\beta_i)$, and let $k = \max\{|\beta_i|\}$. Then

$$\begin{aligned} x &= \sum_{i=1}^n \lambda_i \alpha_i \beta_i^* = \sum_{i=1}^n \lambda_i \alpha_i r(\alpha_i) \beta_i^* \\ &= \sum_{i=1}^n \lambda_i \alpha_i \left(\sum_{\gamma \in X_{r(\alpha_i), k-|\beta_i|}} \gamma \gamma^* \right) \beta_i^* \\ &= \sum_{i=1}^n \sum_{\gamma \in X_{r(\alpha_i), k-|\beta_i|}} \lambda_i \alpha_i \gamma (\beta_i \gamma)^*. \end{aligned}$$

Re-indexing the last sum, we see that

$$x = \sum_{j=1}^l \hat{\lambda}_j \mu_j \nu_j^*,$$

where each μ_j extends an α_i , each ν_j extends a β_i , and all the ν_j have length k or are shorter but end at a sink. In particular, if $\nu_j \neq \nu_i$ then $\nu_j^* \nu_i = 0$.

We are now ready to show that all unitaries in $L_R(E)$ can be written in the form from Lemma 4.2 when R is a unital, commutative subring of \mathbb{C} closed under complex conjugation that has an essentially unique partition of the unit. The following proposition is due to Chris Smith in the case where $R = \mathbb{Z}$.

Proposition 4.4. *Let E be a finite graph, let R be a unital, commutative subring of \mathbb{C} closed under complex conjugation that has an essentially unique partition of the unit, and let $u \in L_R(E)$ be a unitary. Then there exist paths α_i, β_i such that*

$$u = \sum_{i=1}^n \lambda_i \alpha_i \beta_i^*,$$

with

- (1) $\sum_{i=1}^n \alpha_i \alpha_i^* = 1$,
- (2) $\sum_{i=1}^n \beta_i \beta_i^* = 1$, and,

$$(3) \quad |\lambda_i| = 1.$$

Proof. By Remark 4.3,

$$u = \sum_{i=1}^n \lambda_i \alpha_i \beta_i^*$$

for some $\lambda_i \in R$ and paths α_i, β_i with $r(\alpha_i) = r(\beta_i)$ such that the β_i are all of some fixed length t or shorter but ending in a sink. By combining like terms and dropping terms where $\lambda_i = 0$, we may assume that the pairs (α_i, β_i) are distinct and that each $\lambda_i \neq 0$.

We will now show that the β_i are distinct and incidentally that $\lambda_i = \pm 1$ for each i . Fix an index $1 \leq l \leq n$. We have

$$\begin{aligned} r(\beta_l) &= \beta_l^* \beta_l = \beta_l^* u^* u \beta_l = \beta_l^* \left(\left(\sum_{i=1}^n \lambda_i \beta_i \alpha_i^* \right) \left(\sum_{j=1}^n \overline{\lambda_j} \alpha_j \beta_j^* \right) \right) \beta_l \\ &= \beta_l^* \left(\sum_{i,j=1}^n \lambda_i \overline{\lambda_j} \beta_i \alpha_i^* \alpha_j \beta_j^* \right) \beta_l = \sum_{i,j=1}^n \lambda_i \overline{\lambda_j} \beta_l^* \beta_i \alpha_i^* \alpha_j \beta_j^* \beta_l. \end{aligned}$$

By construction of the β_i , we see that $\beta_l^* \beta_j = 0$ if $\beta_i \neq \beta_j$. To ease notation, let $L = \{i \mid \beta_i = \beta_l\}$. Continuing the computation we get

$$(4.1) \quad r(\beta_l) = \sum_{i,j \in L} \lambda_i \overline{\lambda_j} \beta_l^* \beta_l \alpha_i^* \alpha_j \beta_l^* \beta_l = \sum_{i,j \in L} \lambda_i \overline{\lambda_j} \alpha_i^* \alpha_j,$$

where the last equality follows from $\beta_l^* \beta_l = r(\beta_l) = r(\alpha_i)$ for $i \in L$.

Recall that all Leavitt path algebras are \mathbb{Z} -graded, that vertex projections are homogeneous of degree 0, and elements of the form $\mu\nu^*$ are homogeneous of degree $|\mu| - |\nu|$ ([40, Definition 4.5 and Proposition 4.7]). Since the left hand side of (4.1) is homogeneous of degree 0 the right hand side must be so too. Hence, we can discard any element not of degree 0 from the sum. Since the pairs (α_i, β_i) are distinct, and since we only sum over indices i, j with $\beta_i = \beta_l = \beta_j$, we see that the α_i appearing in the sum must be distinct. Thus, for $i \neq j$ the term $\alpha_i^* \alpha_j$ is either 0 or homogeneous of degree $|\alpha_i| - |\alpha_j|$ (note that if $|\alpha_i| = |\alpha_j|$ then $\alpha_i^* \alpha_j = 0$ since the paths are distinct). Thus, we get

$$r(\beta_l) = \sum_{i \in L} |\lambda_i|^2 \alpha_i^* \alpha_i.$$

Using that $\alpha_i^* \alpha_i = r(\alpha_i) = r(\beta_l)$, it follows that

$$r(\beta_l) = \left(\sum_{i \in L} |\lambda_i|^2 \right) r(\beta_l).$$

Since each λ_i is non-zero and R has an essentially unique partition of the unit, this equality can only hold if $|L| = 1$ and $|\lambda_l| = 1$. Hence, all the β_i are distinct and $|\lambda_i| = 1$ for each i , i.e. (3) holds.

Since the β_i are distinct, $\beta_i^* \beta_j = 0$ if $i \neq j$, and as always $\beta_i^* \beta_i = r(\beta_i)$. Thus,

$$\begin{aligned} 1 &= uu^* = \left(\sum_{i=1}^n \lambda_i \alpha_i \beta_i^* \right) \left(\sum_{j=1}^n \overline{\lambda_j} \beta_j \alpha_j^* \right) \\ &= \sum_{i=1}^n |\lambda_i|^2 \alpha_i \alpha_i^* = \sum_{i=1}^n \alpha_i \alpha_i^*. \end{aligned}$$

That is, (1) holds. By Lemma 4.1, (1) implies $\alpha_i^* \alpha_j = 0$ if $i \neq j$, so

$$1 = u^* u = \left(\sum_{j=1}^n \lambda_j \beta_j \alpha_j^* \right) \left(\sum_{i=1}^n \overline{\lambda_i} \alpha_i \beta_i^* \right) = \sum_{i=1}^n \beta_i \beta_i^*.$$

Hence, (2) also holds. \square

In the case of a finite graph with no sinks, we can give a completely geometric description of when a set of paths $\alpha_1, \alpha_2, \dots, \alpha_n$ satisfy that the projections $\alpha_i \alpha_i^*$ sum to 1.

Lemma 4.5. *Let E be a finite graph with no sinks, let R be a unital commutative ring with characteristic 0, and let α_i , $i = 1, 2, \dots, n$, be paths in E . The following are equivalent*

- (1) $\bigsqcup_{i=1}^n \mathcal{Z}(\alpha_i) = E^\infty$.
- (2) $\sum_{i=1}^n \alpha_i \alpha_i^* = 1$.

Proof. By repeated applications of (v) from Definition 2.5, it follows that (1) implies (2). Suppose now that (2) holds. We first show that $\cup_i \mathcal{Z}(\alpha_i) = E^\infty$. Suppose for contradiction that there is some $\mu \in E^\infty$ that is not in $\cup_i \mathcal{Z}(\alpha_i)$, and let ν be the initial segment of μ of length $\max |\alpha_i|$. Then $\alpha_i^* \nu = 0$ for all i and therefore

$$\nu = 1\nu = \left(\sum_{i=1}^n \alpha_i \alpha_i^* \right) \nu = \sum_{i=1}^n \alpha_i \alpha_i^* \nu = 0,$$

a contradiction. By Lemma 4.1, we have $\alpha_i^* \alpha_j = 0$ for $i \neq j$. This implies that no α_i is an initial segment of any α_j , and hence that the cylinder sets are disjoint. \square

5. PROJECTIONS IN $L_R(E)$

The aim of this section is to study the projections in $L_R(E)$ for a finite graph E and R a unital, commutative subring of \mathbb{C} closed under complex conjugation that has an essentially unique partition of the unit. Specifically, it will be proved that all projections in $L_R(E)$ are elements of the diagonal.

Definition 5.1. Let E be a graph. A projection $p \in L_R(E)$ is said to be *diagonal* if $p \in \mathcal{D}(L_R(E))$.

The following Example gives a basic illustration of a Leavitt path algebra where all projections are diagonal.

Example 5.2. Consider the graph



It is well known that $L_{\mathbb{Z}}(F_n) \cong M_n(\mathbb{Z})$, and furthermore that the diagonal is exactly the diagonal matrices. We claim that $L_{\mathbb{Z}}(F_n)$ only has diagonal projections.

Suppose

$$P = \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{pmatrix}$$

is a projection in $M_n(\mathbb{Z})$. The relations $P = P^*$ and $P = P^2$ imply that $P = PP^T$, so for every i we have

$$a_{ii} = \sum_{j=1}^n a_{ij}^2.$$

This can only be satisfied if P is a diagonal matrix with entries in $\{0, 1\}$.

The following two lemmas will pave the way for the proof that all projections in $L_{\mathbb{Z}}(E)$ are diagonal.

Lemma 5.3. *Let E be a graph and let R be a unital, commutative subring of \mathbb{C} closed under complex conjugation that has an essentially unique partition of the unit. If $x, y \in L_R(E)$ satisfy $2x = 2y$ then $x = y$.*

Proof. Let $\iota_E: L_R(E) \rightarrow C^*(E)$ be the embedding from Lemma 3.4. We have

$$2\iota_E(x) = \iota_E(2x) = \iota_E(2y) = 2\iota_E(y),$$

so since 2 is invertible in \mathbb{C} , we get $\iota_E(x) = \iota_E(y)$. Because ι_E is an embedding, $x = y$. \square

Remark 5.4. Let E be a graph, R a commutative unital ring. Using a process called “adding tails” (see [12]), we can find a graph F with no sinks such that $L_R(E)$ embeds into $L_R(F)$ as $*$ -algebras, furthermore this embedding maps E^* into F^* .

For Leavitt path algebras over fields this is proved in [4, Section 5], where infinite emitters are also “desingularized”. We are only interested in removing sinks, so we preform a partial desingularization, we are also interested in working over general rings, not just fields, however the arguments for constructing the embedding given in [4] works perfectly well in this setting, so we only review them briefly. We define F as follows

$$\begin{aligned} F^0 &= E^0 \sqcup \{w_i \mid i \in \mathbb{N}, w \in E^0, w \text{ is a sink}\}, \\ F^1 &= E^1 \sqcup \{e_i^w \mid i \in \mathbb{N}, w \in E^0, w \text{ is a sink}\}, \end{aligned}$$

we let the range and source maps extend those of E and define $r(e_i^w) = e_{i+1}^w$ and

$$s(e_i^w) = \begin{cases} w_{i-1}, & \text{if } i > 1 \\ w, & \text{otherwise} \end{cases}.$$

Now we can define a $*$ -homomorphism $\phi: L_R(E) \rightarrow L_R(F)$ by stipulating that $\phi(v) = v$ and $\phi(e) = e$ for all $v \in E^0$ and $e \in E^1$. By the Graded Uniqueness Theorem ([40, Theorem 5.3]) ϕ is injective.

Lemma 5.5. *Let E be a graph, let R be a unital, commutative subring of \mathbb{C} closed under complex conjugation that has an essentially unique partition of the unit, let $x \in L_R(E)$ and let $\lambda \in R$. If $\alpha, \beta \in E^*$ are such that*

$$2x = \alpha + \lambda\beta,$$

then $\alpha = \beta$.

Proof. By Remark 5.4, we can assume that E has no sinks. We may write

$$x = \sum_{i=1}^n \lambda_i \mu_i \nu_i^*,$$

where $\lambda_i \in R$, $\mu_i, \nu_i \in E^*$. Pick a path γ based at $r(\alpha)$ such that $|\gamma|$ is greater than $\max\{|\nu_i|\}$. Then $x\gamma$ is a polynomial in real edges,

$$x\gamma = \sum_{k=1}^m \hat{\lambda}_k \xi_k,$$

where $\hat{\lambda}_k \in R$ and $\xi_k \in E^*$. Thus,

$$\sum_{k=1}^m 2\hat{\lambda}_k \xi_k = 2x\gamma = \alpha\gamma + \lambda\beta\gamma.$$

The real paths form a linearly independent set in $L_R(E)$ ([40, Proposition 4.9]), so if $\alpha\gamma \neq \beta\gamma$, it follows from the above equation that there is some subset $I \subseteq \{1, 2, \dots, m\}$ such that

$$1 = \sum_{k \in I} 2\hat{\lambda}_k = 2 \left(\sum_{k \in I} \hat{\lambda}_k \right),$$

which contradicts that $\frac{1}{2} \notin R$ (Remark 2.10). Therefore, we must have $\alpha\gamma = \beta\gamma$. Since $s(\gamma) = r(\alpha)$, we have $\alpha\gamma \neq 0$ and hence we must have $\beta\gamma \neq 0$ so $s(\gamma) = r(\beta)$. Thus $\alpha = \beta$. \square

Theorem 5.6. *Let E be a finite graph and let R be a unital, commutative subring of \mathbb{C} closed under complex conjugation that has an essentially unique partition of the unit. If $p \in L_R(E)$ is a projection then*

$$p = \sum_{i=1}^n \beta_i \beta_i^*,$$

for some paths $\beta_i \in E^*$ with $\beta_i^* \beta_j = 0$ for $i \neq j$.

Proof. Put $u = 2p - 1$. Then u is a self-adjoint unitary, i.e. $u = u^*$ and $u^2 = 1$. By Proposition 4.4, we can write

$$u = \sum_{i=1}^n \lambda_i \alpha_i \beta_i^*,$$

where $|\lambda_i| = 1$, $\alpha_i, \beta_i \in E^*$ and $\sum_{i=1}^n \alpha_i \alpha_i^* = 1 = \sum_{i=1}^n \beta_i \beta_i^*$. Thus,

$$2p = 1 + u = \sum_{i=1}^n \beta_i \beta_i^* + \sum_{i=1}^n \lambda_i \alpha_i \beta_i^* = \sum_{i=1}^n (\beta_i + \lambda_i \alpha_i) \beta_i^*.$$

Let $k \in \{1, 2, \dots, n\}$ be given. By Lemma 4.1 $\beta_i^* \beta_k = 0$ when $i \neq k$, so

$$2p\beta_k = \beta_k + \lambda_k \alpha_k.$$

By Lemma 5.5, this implies that $\alpha_k = \beta_k$. So

$$u = \sum_{i=1}^n \lambda_i \beta_i \beta_i^*.$$

We then get

$$\beta_k = (u^2) \beta_k = \left(\sum_{i=1}^n \lambda_i \beta_i \beta_i^* \right) \left(\sum_{j=1}^n \lambda_j \beta_j \beta_j^* \right) \beta_k = \left(\sum_{i=1}^n \lambda_i^2 \beta_i \beta_i^* \right) \beta_k = \lambda_k^2 \beta_k.$$

Since the real paths are linearly independent ([40, Proposition 4.9]), we get that $\lambda_k = \pm 1$. It follows that

$$2p = \sum_{i=1}^n \varepsilon_i \beta_i \beta_i^*,$$

where $\varepsilon_i \in \{0, 2\}$. An application of Lemma 5.3 completes the proof. \square

The following result is an immediate consequence of the preceding theorem.

Corollary 5.7. *Let R be a unital, commutative subring of \mathbb{C} closed under complex conjugation that has an essentially unique partition of the unit. If E is a finite graph, then $L_R(E)$ only has diagonal projections.*

It is not clear that this result only holds when R is a unital, commutative subring of \mathbb{C} closed under complex conjugation that has an essentially unique partition of the unit, however it is worth noting that the result certainly requires some restriction on the ring of coefficients. For instance,

$$p = \frac{1}{2}(aa^* + ab^* + ba^* + bb^*)$$

is non-diagonal projection in $L_{2,\mathbb{C}}$.

6. CONCLUSIONS

Proposition 6.1. *Let E, F be graphs, let R be a subring of \mathbb{C} closed under complex conjugation, and let $\phi: L_R(E) \rightarrow L_R(F)$ be a $*$ -homomorphism. If $L_R(F)$ only has diagonal projections, then $\phi(\mathcal{D}(L_R(E))) \subseteq \mathcal{D}(L_R(F))$.*

Proof. Let $\alpha \in E^*$ be given. Since $\alpha\alpha^*$ is a projection in $L_R(E)$ and since ϕ is a $*$ -homomorphism, $\phi(\alpha\alpha^*)$ is a projection. By the assumption on $L_R(F)$, it follows that $\phi(\alpha\alpha^*) \in \mathcal{D}(L_R(F))$. By linearity of ϕ and the fact that $\mathcal{D}(L_R(F))$ is an algebra, we get that $\phi(\mathcal{D}(L_R(E))) \subseteq \mathcal{D}(L_R(F))$. \square

Remark 6.2. The requirement that $L_R(F)$ only has diagonal projections is clearly crucial for the result. For instance, there exist $*$ -automorphisms of $L_{2,\mathbb{C}}$ that do not preserve the diagonal. As an example of this, consider the unitary

$$u = \frac{1}{\sqrt{2}}(aa^* - ab^* + ba^* + bb^*),$$

and define an automorphism ψ of $L_{2,\mathbb{C}}$ by $\psi(x) = uxu^*$. Then

$$\psi(aa^*) = \frac{1}{2}(aa^* + ab^* + ba^* + bb^*),$$

so ψ does not preserve the diagonal. This u was constructed by mapping the unitary that rotates by 45 degrees from $M_2(\mathbb{C})$ into $L_{2,\mathbb{C}}$ using the map discussed in [15, Example 5.13].

Theorem 6.3. *Let E, F be finite, essential, strongly connected graphs that satisfy Condition (L), and let R be a subring of \mathbb{C} closed under complex conjugation. Assume furthermore that $L_R(F)$ only has diagonal projections. If $L_R(E)$ is $*$ -isomorphic to $L_R(F)$ then*

$$\text{sgn}(\det(I - A_E)) = \text{sgn}(\det(I - A_F)).$$

Proof. Suppose $\phi: L_R(E) \rightarrow L_R(F)$ is a $*$ -isomorphism. By Proposition 6.1, we must have $\phi(\mathcal{D}(L_R(E))) \subseteq \mathcal{D}(L_R(F))$, so by Theorem 3.6,

$$\text{sgn}(\det(I - A_E)) = \text{sgn}(\det(I - A_F)). \quad \square$$

Corollary 6.4. *Let E, F be finite, essential, strongly connected graphs that satisfy Condition (L) and let R be a unital, commutative subring of \mathbb{C} closed under complex conjugation that has an essentially unique partition of the unit. If $L_R(E)$ is $*$ -isomorphic to $L_R(F)$ then*

$$\text{sgn}(\det(I - A_E)) = \text{sgn}(\det(I - A_F)).$$

Proof. This is an immediate consequence of Theorem 6.3, since all projections in $L_R(E)$ are diagonal by Corollary 5.7. \square

Corollary 6.5. *Let R be a unital, commutative subring of \mathbb{C} closed under complex conjugation that has an essentially unique partition of the unit. Then $L_{2,R}$ is not $*$ -isomorphic to $L_{2-,R}$. In particular, $L_{2,\mathbb{Z}}$ is not $*$ -isomorphic to $L_{2-, \mathbb{Z}}$.*

Proof. Recall that by definition $L_{2,R} = L_R(E_2)$ and $L_{2-,R} = L_R(E_{2-})$. We see that the adjacency matrices are

$$A_{E_2} = \begin{pmatrix} 2 \end{pmatrix},$$

and

$$A_{E_{2-}} = \begin{pmatrix} 2 & 1 & 0 \\ 1 & 1 & 1 \\ 0 & 1 & 1 \end{pmatrix}.$$

Hence, $\det(I - A_{E_2}) = -1$ and $\det(I - A_{E_{2-}}) = 1$, so by Corollary 6.4, $L_{2,\mathbb{Z}} \not\cong L_{2-,\mathbb{Z}}$ as $*$ -algebras. \square

Although Corollary 6.5 does not apply to fields, we can use it to say something about the situation for fields.

Remark 6.6. Suppose K is a field with characteristic 0. Then $L_{2,\mathbb{Z}}$ will embed into $L_{2,K}$, and $L_{2-,\mathbb{Z}}$ will embed into $L_{2-,K}$. Assume that there exists a $*$ -isomorphism $\phi: L_{2-,K} \rightarrow L_{2,K}$ which restricts to a $*$ -homomorphism from $L_{2-,\mathbb{Z}}$ to $L_{2,\mathbb{Z}}$. The restriction is also injective, so by Corollary 6.5, it cannot be surjective.

One now wonders how common it is for $*$ -isomorphisms between Leavitt path algebras over K to restrict to $*$ -homomorphisms between the corresponding algebras over \mathbb{Z} . In general, homomorphisms will not restrict in this way. For instance, each $z \in \mathbb{C}$ of modulus one defines a *gauge automorphism* $\gamma_z: L_{2,\mathbb{C}} \rightarrow L_{2,\mathbb{C}}$ given by $\gamma_z(a) = za$ and $\gamma_z(b) = zb$ which clearly does not restrict to an automorphism of $L_{2,\mathbb{Z}}$. However, in many cases where homomorphisms are written down explicitly, the field of coefficients has not been used, and such homomorphisms must clearly restrict to homomorphisms over \mathbb{Z} . For some examples of this phenomenon, see for instance [2, 5, 15, 37]. In fact, at the end of [40], Tomforde suggests that Leavitt path algebras over \mathbb{Z} may provide the key to understanding this non-dependence on the field of coefficients.

Our result leaves two ways for $L_{2-,K}$ and $L_{2,K}$ to be $*$ -isomorphic: Either the isomorphism makes explicit use of the field, so that it does not induce a $*$ -homomorphism from $L_{2-,\mathbb{Z}}$ to $L_{2,\mathbb{Z}}$. Or, if the isomorphism does induce such a $*$ -homomorphism, then this induced map cannot be surjective. Either way, a $*$ -isomorphism between $L_{2-,K}$ and $L_{2,K}$, if it exists, will have to be a fairly complicated map.

We conclude with a few remarks on the possibility that $L_{2,R}$ and $L_{2-,R}$ are Morita equivalent.

Remark 6.7. Let K be a field. If $L_{2,K}$ and $L_{2-,K}$ are Morita equivalent, they must be isomorphic as rings (this follows from the argument given in the proof of (2) implies (1) in [37, Proposition 10.4]). It is also worth noting that there are no known examples of Leavitt path algebras that are isomorphic as rings, but not as $*$ -algebras. See [6] for a discussion of how various notions of isomorphism of Leavitt path algebras are related to isomorphism of graph C^* -algebras.

When working over a commutative, unital ring R , the connection between Morita equivalence and ring isomorphism of $L_{2,R}$ and $L_{2-,R}$ is less clear. If R is a regular supercoherent ring, then both $L_{2,R}$ and $L_{2-,R}$ have trivial algebraic K -theory [8]. So at least this does not provide an obvious proof that they are not Morita equivalent. We note that the class of regular supercoherent rings covers the class of Noetherian regular rings which in turn contains the class of principal ideal domains.

To go from a Morita equivalence of $L_{2,K}$ and $L_{2-,K}$ to a ring isomorphism, one uses that $L_{2,K}$ is simple. In particular, the description of the K_0 -group of a simple purely infinite ring given in [9, Corollary 2.2] is crucial. However, if the ring R has

non-trivial ideals, then $L_{2,R}$ will clearly not be simple. Therefore, it is unclear to the authors if Morita equivalence of $L_{2,R}$ and $L_{2-,R}$ implies ring isomorphism.

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